

# Additivity property and emergence of power laws in nonequilibrium steady states

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We show that an equilibrium-like additivity property can remarkably lead to power-law distributions observed frequently in a wide class of out-of-equilibrium systems. The additivity property can determine the full scaling form of the distribution functions and the associated exponents. The asymptotic behaviour of these distributions is solely governed by branch-cut singularity in the variance of subsystem mass. To substantiate these claims, we explicitly calculate, using the additivity property, subsystem mass distributions in a wide class of previously studied mass aggregation models as well as in their variants. These results could help in thermodynamic characterization of nonequilibrium critical phenomena.

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## I. INTRODUCTION

Simple power-law scaling is ubiquitous in nature [1]. They appear in the distributions of drainage area of rivers [2], droplet size [3, 4], size of clusters formed in polymerization processes [5], rain size [6], size of fragments in fractured solids [7], population and wealth [8, 9], and in stock market fluctuations [10], etc. Evidently, power laws, which are usually associated with criticality through emergence of a diverging length scale, are observed in widely unrelated systems, suggesting existence of some broad underlying principle. Recent evidence that living systems might be operating, independent of most of the microscopic details, in the vicinity of a critical regime [11] indeed invoke further questions - how and why systems adapt to near-criticality.

There have been several attempts to reveal the origin of the power laws in nature, through studies of paradigmatic nonequilibrium models - most appealing being sandpile [12–14] and mass aggregation models [15–20]. Many of these models - where there is a conservation law or, in case of violation, the law is weakly violated in the sense that the systems are slowly driven - are intimately connected to each other. For example, the mass aggregation models [17–19, 21] are connected to directed abelian sandpile model [22] or to the models of river network [2].

In this paper, we argue that power-law distributions in out-of-equilibrium systems can arise simply from additivity property, the tenet of equilibrium thermodynamics. We find that the divergence in the response function is the key: Diverging fluctuations can, in principle, arise from distributions other than power laws, which are however prohibited if one imposes additivity and consequent fluctuation-response (FR) relation. The response function determines the full scaling form of the distribution, *at as well as away* from criticality, and critical exponents

originate from the singularity in the response function. To demonstrate this, we consider mass aggregation models which are known to have nonequilibrium steady state with scale invariant structures. At *all* mass densities, the distribution function  $P_v(m)$  of mass  $m$  in a subsystem of volume  $v$ , which is obtained solely from the FR relation, is shown to have a scaling form  $P_v(m) \sim m^{-\tau} \exp(\tilde{\mu}m)$ . The quantity  $\tilde{\mu}(\rho) = \mu(\rho) - \mu(\rho_c)$ , inverse of a cut-off mass  $m^*(\rho) = -1/\tilde{\mu}(\rho)$ , is an analogue of equilibrium-like chemical potential and provides a useful thermodynamic interpretation of the emergence of power laws in nonequilibrium steady states. The exponent  $\tau$  and the critical properties of chemical  $\mu(\rho)$  arise from a multiple-pole or branch-cut singularity in the variance at a critical mass density  $\rho_c$ . As the critical density is approached  $\rho \rightarrow \rho_c$ , nonequilibrium chemical potential vanishes  $\mu(\rho) \rightarrow 0$ , leading to pure power laws. Beyond the critical density  $\rho > \rho_c$ , there is a gas-liquid like phase coexistence.

The above result immediately provides answer to why  $m^{-5/2}$  power law, at or away from criticality, appears so often in mass aggregation models - especially in higher dimensions, at all densities and irrespective of that the motion of the diffusing masses is biased or not [19, 21, 23, 24]. Interestingly, the same power law appears in  $k$ -mer distribution in the classic Flory-Stockmayer [25] theory of polymerization and also in particle number distribution in three dimensional ideal Bose gas near critical point, irrespective of whether the systems are in or out of equilibrium - thus indicating a *universality*. We demonstrate that the  $m^{-5/2}$  law is a consequence of a simple-pole singularity in the variance. The whole analysis is extended also to nonconserved mass aggregation models. We validate our theory by explicitly calculating mass distributions in previously studied mass aggregation models and their variants and by comparing them with simulations.

Organization of the paper is as follows. In section II.A, we discuss additivity property; in section II.B, we discuss the connection between singularity in the variance and the asymptotic behaviour of the mass distribution func-

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tion. In section III, we illustrate our analytic methods in a broad class of model systems, both in conserved-mass aggregation models and its nonconserved versions. Finally, we summarize our results with a concluding perspective.

## II. THEORY

### A. Additivity Property

We start by invoking an additivity property which a wide class of systems, irrespective of that they are in or out of equilibrium, could possess. Consider a continuous-mass (generalization to discrete masses is straightforward) transport process on a lattice of size  $V$  and then divide the system in  $\nu = V/v$  identical subsystems or cells,  $k$ th subsystem with mass  $m_k$ , where total mass  $M = \sum_k m_k$  remains conserved. Provided the subsystems are large compared to spatial correlation length, they could be considered statistically almost independent [27–30]. In that case, the joint subsystem mass distribution in the steady state can be written in a product form,

$$\mathcal{P}[\{m_k\}] \simeq \frac{\prod_{k=1}^{\nu} w_v(m_k)}{Z(M, V)} \delta\left(\sum_k m_k - M\right), \quad (1)$$

where  $w_v(m_k)$  is an unknown weight factor (to be determined later; see Eq. 13) depending only on the subsystem mass  $m_k$ ,  $Z = \prod_k \int dm_k w_v(m_k) \delta(\sum_k m_k - M) \equiv \exp[-Vf(\rho)]$  the partition sum,  $f(\rho)$  a nonequilibrium free energy density and  $\rho = M/V$  mass density. The product form in Eq. 1 amounts to an equilibrium-like additivity property, in the sense that a free energy function  $F = \sum_k \ln w_v(m_k)$  is minimized in the macrostate.

Using standard statistical mechanics [26], Eq. 1 leads to the probability distribution  $\text{Prob}[m_k \in (m, m+dm)] = P_v(m)dm$  for subsystem mass where

$$P_v(m) = \frac{w_v(m)e^{\mu m}}{\mathcal{Z}} \quad (2)$$

with  $\mu(\rho)$  a nonequilibrium chemical potential, and  $\mathcal{Z}$  the normalization constant. The weight factor  $w_v(m)$  and chemical potential  $\mu(\rho) = df/d\rho$  can be obtained using a fluctuation-response relation [27–31],

$$\frac{d\rho}{d\mu} = \sigma^2(\rho), \quad (3)$$

where the scaled variance  $\sigma^2(\rho) = (1/v)(\langle m_k^2 \rangle - \langle m_k \rangle^2)$  in the limit of  $v \gg 1$ . The free energy density function  $f(\rho)$  can be obtained through the relation  $\mu(\rho) = df/d\rho$ , i.e.,  $f(\rho) = \int \mu(\rho)d\rho + \beta$  with chemical potential  $\mu(\rho) = \int 1/\sigma^2(\rho)d\rho + \alpha$  (obtained from Eq. 3) and  $\alpha$  and  $\beta$  arbitrary constants of integration. Then Laplace transform of  $w_v(m)$  is written as  $\tilde{w}_v(s) = \int w_v(m) \exp(-sm)dm \equiv e^{-\lambda_v(s)}$ , i.e.,

$$e^{-\lambda_v(s)} = \int w_v(m) e^{-sm} dm. \quad (4)$$

Then, the function  $\lambda_v(s)$  can be obtained from Legendre transform of free energy density function  $f(\rho)$  [32],

$$\lambda_v(s) = v[\inf_{\rho}\{f(\rho) + s\rho\}] = v[f(\rho^*) + s\rho^*], \quad (5)$$

where  $\rho^*(s)$  is the solution of

$$s = -\mu(\rho^*). \quad (6)$$

As discussed later, Eq. 5 requires concavity and differentiability of  $f(\rho)$ . In the discrete case, the weight factor can be calculated as  $w_v(m) = (1/2\pi i) \int_C \tilde{w}_v(z)/z^{m+1} dz$  where  $\tilde{w}_v(z) = \sum_{m=0}^{\infty} z^m w_v(m)$  is obtained from  $\tilde{w}(s)$  by substituting  $s = -\ln z$  with  $C$  a suitably chosen contour in the complex  $z$ -plane.

### B. Singularity in Variance and Mass Distribution

Importantly, the distribution function  $P_v(m)$  is determined solely by the functional form of the scaled variance  $\sigma^2(\rho)$ . We argue below that singular response functions generate only power-law distributions. Other functional form of mass distribution  $P_v(m)$  with diverging variance is also possible [33], which, we show, however are not allowed if the FR relation holds. In this paper, we mainly focus on multi-pole singularity at a finite density  $\rho_c$ ,

$$\sigma^2(\rho) = \begin{cases} \frac{g(\rho)}{(\rho_c - \rho)^n} & \text{for } \rho < \rho_c, \\ \infty & \text{otherwise.} \end{cases} \quad (7)$$

This form, with  $0 < n < \infty$ , is relevant in the context of a wide class of mass aggregation models as discussed in section III. The analytic part  $g(\rho)$  is not particularly relevant in determining the asymptotic form of the distribution  $P_v(m)$ , however it contributes to the exact form of  $P_v(m)$  (discussed in section II.B.IV). In fact, other kinds of singularities, such as logarithmic singularity  $\sigma^2(\rho) \sim [\ln(\rho_c - \rho)]^p$  or exponential singularity  $\exp[(\rho_c - \rho)^{-p}]$  where  $p > 0$ ,  $1/|\rho - \rho_c|^n$ , and the case with  $n < 0$  can also arise. One can show that they all lead to power laws, possibly with logarithmic corrections to the power-law scaling (discussed in the following sections).

The divergence in the variance, as in Eq. 7 (or in the cases of logarithmic and exponential divergence), indeed has broad implications, not only in conserved-mass aggregation models but also in their nonconserved versions. Note that the FR relation in Eq. 3 implies that free energy density  $f(\rho)$  is not a strictly concave function of  $\rho$  and has a linear branch of slope  $\mu(\rho_c)$  for  $\rho \geq \rho_c$ . Moreover,  $f''(\rho = \rho_c) = \mu'(\rho = \rho_c) = 0$  (prime denotes derivative w.r.t.  $\rho$ ) implies a point of inflection in  $f - \rho$  curve at  $\rho = \rho_c$ . That is, free energy density function can be written as

$$f(x) = \begin{cases} \int \mu(x) dx & \text{for } x < \rho_c, \\ \mu(\rho_c)(x - \rho_c) + f(\rho_c) & \text{otherwise.} \end{cases} \quad (8)$$

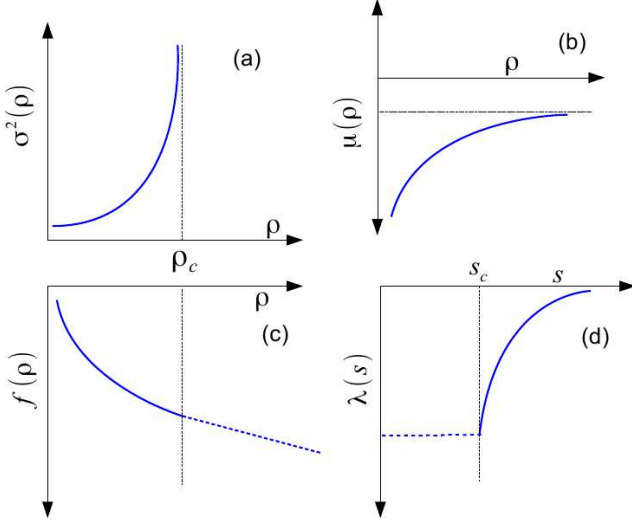


FIG. 1: (Color online) Schematic representation of condensation transition. Panel (a): variance  $\sigma^2(\rho)$  as a function of density  $\rho$ ; panel (b): chemical potential  $\mu(\rho)$  as a function of  $\rho$ ; panel (c): free energy density  $f(\rho)$  as a function of  $\rho$ ; panel (d): Legendre transform  $\lambda(s)$  of free energy density as a function of  $s$ .

Consequently, Legendre transform of  $f(\rho)$  develops a branch-cut singularity (see Eq. 11); for schematic representation of the above analysis, see Fig. 1. This construction of a nonequilibrium free energy function  $f(\rho)$  from a general thermodynamic consideration readily explains the phase coexistence between a fluid and a condensate, as observed in the past in many out-of-equilibrium systems (discussed in section III).

### 1. Multi-pole singularity

To analyse the behaviour of  $\lambda_v(s)$  in the case of Eq. 7, we integrate Eq. 3 near  $\rho = \rho_c$  and obtain

$$\mu(\rho) \simeq -\frac{(\rho_c - \rho)^{n+1}}{(n+1)g(\rho_c)} [1 + \mathcal{O}(\rho - \rho_c)] + \alpha, \quad (9)$$

which gives  $(\rho_c - \rho^*) \simeq [(n+1)g(\rho_c)(s + \alpha)]^{1/(1+n)}$  by using Eqs. 6 and 9. Integrating chemical potential  $\mu(\rho)$ , we get free energy function

$$f(\rho) \simeq \frac{(\rho_c - \rho)^{n+2}}{(n+1)(n+2)g(\rho_c)} + \alpha\rho + \beta \quad (10)$$

and write  $\lambda_v(s) = v[f(\rho^*) + s\rho^*]$ , in leading order, as

$$\lambda_v(s) \simeq v[a_0 + a_1(s + \alpha) + a_2(s + \alpha)^{\frac{n+2}{n+1}}], \quad (11)$$

where  $a_0, a_1, a_2$  are constants. Thus, we obtain  $\tilde{w}(s) = \exp[-\lambda_v(s)] \simeq \text{const.} \times [1 - va_1(s + \alpha) - va_2(s +$

$\alpha)^{1+1/(1+n)}]$  in leading orders of  $(s + \alpha)$ , implying

$$w_v(m) \sim \frac{e^{-\alpha m}}{m^\tau},$$

where, for large subsystem masses  $m \gg v$ , the power-law exponent  $\tau$  in the denominator is given by

$$\tau = \left[ 2 + \frac{1}{(1+n)} \right], \quad (12)$$

with the following inequality  $2 < \tau < 3$  (since  $0 < n < \infty$  in Eq. 7). This translates into the mass distribution having a scaling form,

$$P_v(m) \propto \frac{1}{m^\tau} e^{\tilde{\mu}(\rho)m} \equiv \frac{1}{(m^*)^\tau} \Phi\left(\frac{m}{m^*}\right), \quad (13)$$

where  $\tilde{\mu}(\rho) = \int_{\rho_c}^{\rho} 1/\sigma^2(\rho)d\rho = \mu(\rho) - \mu(\rho_c)$  is an effective chemical potential, inverse of which gives a cut-off  $m^* = -1/\tilde{\mu}$  in the distribution, and the scaling function  $\Phi(x) = x^{-\tau} \exp(-x)$ . Later, we explicitly calculate  $\tilde{\mu}(\rho)$  in specific model systems. Note that  $\tilde{\mu}(\rho_c) = 0$  at critical point  $\rho = \rho_c$  and consequently  $P_v(m)$  becomes a pure power law. Moreover, by defining a critical exponent  $\delta = 1 + n$  as  $\tilde{\mu}(\rho) \sim (\rho_c - \rho)^\delta$ , we get a scaling relation  $\delta(\tau - 2) = 1$ .

### 2. Logarithmic singularity

Now, we consider the case of logarithmic singularity where variance  $\sigma^2(\rho)$  diverges logarithmically as given below,

$$\sigma^2(\rho) = \begin{cases} g(\rho)[\ln(\rho_c - \rho)]^p & \text{for } \rho < \rho_c, \\ \infty & \text{otherwise.} \end{cases} \quad (14)$$

Integrating Eq. 3 near  $\rho = \rho_c$ , we obtain chemical potential, in leading order of  $(\rho_c - \rho)$ ,

$$\mu(\rho) \simeq -\frac{(\rho_c - \rho)}{g(\rho_c)[\ln(\rho_c - \rho)]^p} + \alpha, \quad (15)$$

which gives  $(\rho_c - \rho^*) \simeq g(\rho_c)[\ln(\rho_c - \rho^*)]^p(s + \alpha)$  from Eq. 6. Free energy density is obtained by integrating above chemical potential  $\mu(\rho)$ ,

$$f(\rho) \simeq -\frac{(\rho_c - \rho)^2}{2g(\rho_c)[\ln(\rho_c - \rho)]^p} + \alpha\rho + \beta, \quad (16)$$

and, accordingly, its Legendre transform, in leading orders,

$$\lambda_v(s) \simeq v[a_0 + a_1(s + \alpha) + a_2(s + \alpha)^2 \{\ln(s + \alpha)\}^p]. \quad (17)$$

For large mass  $m \gg v$ , this implies that the weight factor has a functional form of a power law with logarithmic correction,

$$w_v(m) \sim \frac{(\ln m)^{p-1}}{m^3} e^{-\alpha m},$$

and the corresponding mass distribution function,

$$P_v(m) \propto \frac{(\ln m)^{p-1}}{m^3} e^{\tilde{\mu}(\rho)m}, \quad (18)$$

where effective chemical potential  $\tilde{\mu}(\rho) = \mu(\rho) - \mu(\rho_c)$  and the power-law exponent in the denominator is  $\tau = 3$ , the borderline case of Eqs. 7 and 12 with  $n = 0$ .

### 3. Exponential singularity

We also consider the case where the variance diverges exponentially,  $\sigma^2(\rho) \sim \exp(\rho_c - \rho)^{-p}$  for  $\rho < \rho_c$  and  $\sigma^2(\rho) = \infty$  otherwise. The analysis is similar to the ones given above. Substituting chemical potential  $\mu(\rho) \simeq \text{const} \times (\rho_c - \rho)^{p+1} \exp[-(\rho_c - \rho)^{-p}] + \alpha$  in Eq. 6 and solving in leading order of  $(s + \alpha)$ , we get  $(\rho_c - \rho^*) \sim \{\ln(s + \alpha)\}^{-1/p}$  and consequently

$$\lambda_v(s) \simeq v \left[ a_0 + a_1(s + \alpha) + a_2(s + \alpha) \{\ln(s + \alpha)\}^{-1/p} \right]. \quad (19)$$

For large mass  $m \gg v$ , this leads to the mass distribution function

$$P_v(m) \propto \frac{(\ln m)^{-1-1/p}}{m^2} e^{\tilde{\mu}(\rho)m}, \quad (20)$$

where  $\tilde{\mu}(\rho) = \mu(\rho) - \mu(\rho_c)$  and the power-law exponent in the denominator is  $\tau = 2$ , the borderline case of Eqs. 7 and 12 with  $n = \infty$ .

### 4. Subsystem mass distribution

For any finite  $v$ , it is not easy to find the distribution function  $P_v(m)$  at small or intermediate values of masses  $m \sim v$  because, in that case, one requires to invert Eq. 4 using inverse Laplace transform, i.e., by evaluating the integral,

$$w_v(m) = \frac{1}{2\pi i} \int_C e^{-\lambda_v(s) + ms} ds, \quad (21)$$

on the complex  $s$ -plane; the contour  $C$  on the complex plane should be chosen such that the integral converges. However, in the models we consider here, it is not possible to get an exact functional form of  $\lambda_v(s)$  for all  $s$ , which actually involves solving the transcendental Eq. 6. However, for large subsystem size  $v \gg 1$ , the analysis simplifies as the function  $-(1/v) \ln w_v(m)$  is simply related to  $\lambda_v(s)/v$  by Legendre transformation [32] and therefore, in leading order of  $m$ , is the free energy density function  $f(m/v)$  itself, which has been already constructed in Eq. 8 (see Fig. 1 and related discussions). The subsystem mass distribution function now can be written as

$$P_v(m) \propto \frac{e^{-vf(m/v) + \mu(\rho)m}}{(a + m)^{2.5}}, \quad (22)$$

where the denominator is essentially a sub-leading correction to the free energy function, with  $a \sim \mathcal{O}(v)$  being a model-dependent cut-off mass. The correction term is obtained from the fact that, for large masses  $m \gg v$ , free energy function  $f(x)$  has a linear branch (see Eq. 8) and the mass distribution function must have the asymptotic form  $P_v(m) \sim m^{-\tau} \exp(\tilde{\mu}m)$  as in Eq. 13.

## III. MODELS AND ILLUSTRATION

### A. Conserved Mass Aggregation Models (CMAM)

We now substantiate the above claims in a broad class of nonequilibrium models which were studied intensively in the last couple of decades. In this paper, we mainly focus on the models having multi-pole singularity in the variance. Other singularities, e.g., exponential or logarithmic, are possible, however not as common as the multi-pole one. For the purpose of illustrations, we specifically consider the case with  $n = 1$  and mass distribution at a single site, i.e.,  $v = 1$ . We define a generalized version of conserved mass aggregation models (CMAM) studied in [19–21, 35], for simplicity on a one dimensional lattice of  $L$  sites. We mainly focus on symmetric mass transfer, i.e., masses are transferred symmetrically to either of the neighbours. Here masses (or particles) diffuse, fragment and coalesce stochastically with either of the nearest-neighbour masses according to the following dynamical rules: (1) diffusion of mass  $m_i$  at site  $i$  to  $i \pm 1$  with mass-dependent rate  $D(m_i)$  where  $m_i \rightarrow 0$  and  $m_{i \pm 1} \rightarrow m_{i \pm 1} + m_i$  and (2) fragmentation of a discrete mass  $\Delta$  at site  $i$ , provided  $\Delta \leq m_i$ , and coalescence of the mass to either of the sites  $i \pm 1$  with mass-dependent rate  $w(\Delta)$  where  $m_i \rightarrow m_i - \Delta$  and  $m_{i \pm 1} \rightarrow m_{i \pm 1} + \Delta$  with  $\Delta = 1, 2, \dots$  (continuous  $\Delta$  considered later). Total mass  $M = \sum_{i=1}^V m_i$  is conserved in this process. Even for these simple dynamical rules, the steady state weight in general is not exactly known. However, as spatial correlations are small, the additivity property as in Eq. 1, to a good approximation, is expected to hold.

We calculate the variance  $\sigma^2(\rho)$  of mass at a single site in various special cases, using the additivity property Eq. 1 with  $v = 1$ . We take diffusion rate  $D(m_i) = 1$ , independent of mass  $m_i$ ,  $w(\Delta = 1) = w_1$  (rate of single particle chipping),  $w(\Delta = m_i - 1) = w_2$  (rate of all-but-one particle chipping) and  $w(\Delta) = 0$  otherwise.

#### 1. Case I: CMAM with $w_1 = 1, w_2 = 0$

For  $w_1 = 1$  and  $w_2 = 0$  and symmetric mass transfer, the model becomes the symmetric one studied in [19, 21] (our model is a variant of those studied in [15, 16]). For  $\rho \leq \rho_c$ , using additivity property, we exactly calculated the variance and consequently chemical potential with the critical density  $\rho_c = \sqrt{2} - 1$ . We can calculate the



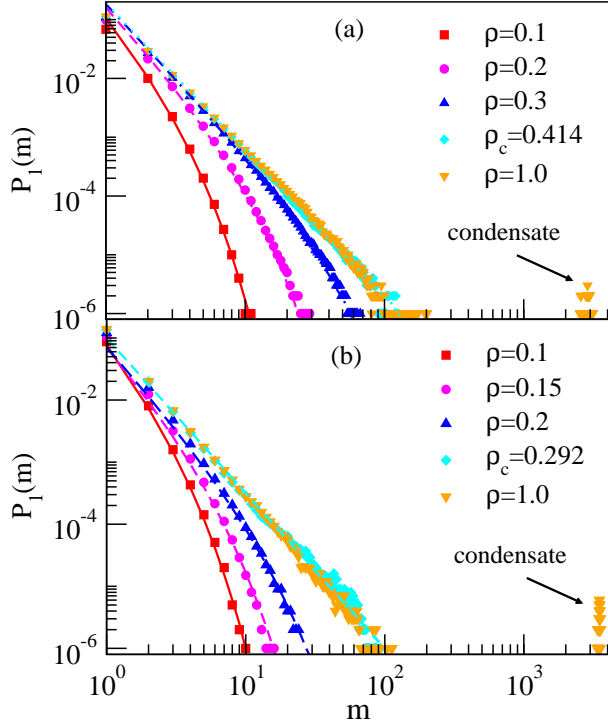


FIG. 2: (Color online) Single-site ( $v = 1$ ) mass distribution functions  $P_1(m)$  (points - simulations) in conserved-mass aggregation models (CMAMs) is compared with analytic expression in Eq. 13 (lines - theory) for various densities. Panel (a): mass chipping rates  $w_1 = 1$ ,  $w_2 = 0$ ; panel (b): mass chipping rates  $w_1 = 0$ ,  $w_2 = 1$ . In all cases, mass diffusion rate  $D = 1$  and system size  $L = 5000$ .

variance as given below (for details, see Appendix A)

$$\begin{aligned}\sigma^2(\rho) &= \frac{\rho(1+\rho)(1+\rho^2)}{(1-2\rho-\rho^2)} \\ &= \frac{\rho(1+\rho)(1+\rho^2)}{(\rho_c - \rho)(\sqrt{2} + 1 + \rho)},\end{aligned}\quad (23)$$

with  $\rho_c = (\sqrt{2} - 1)$ , for which one can obtain a chemical potential  $\mu(\rho)$  and free energy function  $f(\rho)$ , by integrating the fluctuation-response relation as in Eq.2 in the main text,

$$\mu(\rho) = \int \frac{1}{\sigma^2(\rho)} d\rho = -2 \tan^{-1} \rho + \ln \left( \frac{\rho}{1+\rho} \right) + \alpha \quad (24)$$

and, upon one more integration,

$$\begin{aligned}f(\rho) = \int \mu(\rho) d\rho &= -2\rho \tan^{-1} \rho + \rho \ln \left( \frac{\rho}{1+\rho} \right) \\ &\quad - \ln \left( \frac{1+\rho}{1+\rho^2} \right) + \alpha\rho + \beta\end{aligned}\quad (25)$$

where  $\alpha$  and  $\beta$  are two arbitrary constants of integration. For large mass  $m \gg 1$ , the mass distribution function is

calculated to be  $P_1(m) \propto m^{-5/2} \exp[(\mu(\rho) - \mu(\rho_c))m]$ . (for details, see Appendix B).

In panel (a) of Fig. 2, we have plotted single-site ( $v = 1$ ) mass distribution function  $P_1(m)$ , obtained from simulations, as a function of mass  $m$  for various values of densities  $\rho = 0.1, 0.2, 0.3, 0.414$  and  $1.0$  and compare them with the theoretical expression in Eq. 13. The theoretical results have a quite good agreement with the simulation results, especially at large mass  $m \gg 1$ . In panel (a) of Fig. 3, we have plotted subsystem mass distribution function  $P_v(m)$ , with  $v = 100$ , for densities  $\rho = 0.1$  and  $0.2$  and compare them with the theoretical expression in Eq. 22 with cut-off mass  $a \approx 20$ ; agreement between simulations and theory is reasonably good.

## 2. Case II: CMAM with $w_1 = 0, w_2 = 1$

The CMAM with  $w_1 = 0, w_2 = 1$  is a variant of the models studied in [20]. In this case, for  $\rho \leq \rho_c$ , the variance and chemical potential can be exactly obtained using additivity property (for details, see Appendix A). The variance is given by

$$\begin{aligned}\sigma^2(\rho) &= \frac{\rho(1-\rho)(2\rho^2 - 2\rho + 1)}{2\rho^2 - 4\rho + 1} \\ &= \frac{\rho(1-\rho)(2\rho^2 - 2\rho + 1)}{(\rho_c - \rho)(2 + \sqrt{2} - 2\rho)}.\end{aligned}\quad (26)$$

There is a simple pole at the critical density  $\rho_c = 1 - 1/\sqrt{2}$ . By integrating fluctuation-response relation Eq. 3, we get chemical potential

$$\mu(\rho) = 2 \tan^{-1}(1 - 2\rho) - \ln \left[ \frac{1}{2\rho(1-\rho)} - 1 \right] + \alpha, \quad (27)$$

and then free energy density

$$\begin{aligned}f(\rho) &= 2\rho \tan^{-1}(1 - 2\rho) - \ln(1 - \rho) + \ln(1 - 2\rho + 2\rho^2) \\ &\quad - \rho \ln \left[ \frac{1}{2\rho(1-\rho)} - 1 \right] + \alpha\rho + \beta\end{aligned}\quad (28)$$

In panel (b) of Fig. 2, we have plotted single-site ( $v = 1$ ) mass distribution function  $P_1(m)$ , obtained from simulations, for various values of densities  $\rho = 0.1, 0.15, 0.2, 0.292$  and  $1.0$ . We find that the simulation results agree remarkably well with the analytical scaling form  $P_1(m) \propto m^{-5/2} \exp[(\mu(\rho) - \mu(\rho_c))m]$  as in Eq. 13 with  $\tau = 5/2$  (for details of the derivation, see Appendix B). In panel (b) of Fig. 3, we have plotted subsystem mass distribution function  $P_v(m)$  for  $v = 100$  for densities  $\rho = 0.1$  and  $0.15$  and compared them with theory, Eq. 22 with  $a \approx 25$ ; agreement between simulations and theory is reasonably good.

## 3. Other Cases

We have also studied, through simulations, various other cases (with  $D = 1$ ): Case III. -  $w(\Delta = 1) = w_1$ ,

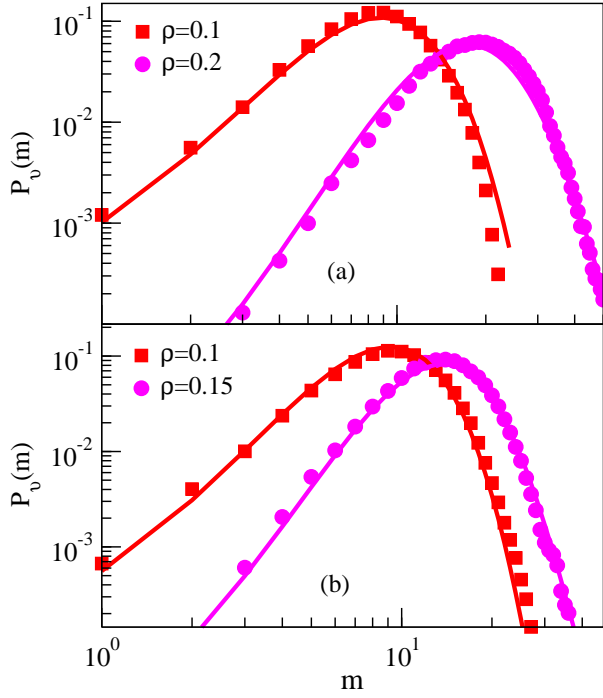


FIG. 3: (Color online) Subsystem mass distribution functions  $P_v(m)$  (points - simulations) in conserved mass aggregation models (CMAMs) is compared with analytic expression in Eq. 22 (lines - theory) for various densities. Panel (a): mass chipping rates  $w_1 = 1$ ,  $w_2 = 0$  and  $a \approx 20$ ; panel (b): mass chipping rates  $w_1 = 0$ ,  $w_2 = 1$  and cut-off mass  $a \approx 25$ . In all cases, mass diffusion rate  $D = 1$ , system size  $L = 10^5$  and subsystem size  $v = 100$ .

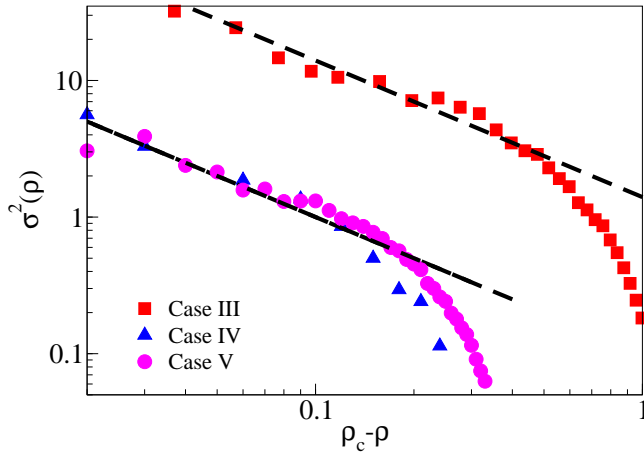


FIG. 4: (Color online) Variance  $\sigma^2(\rho)$  vs.  $(\rho_c - \rho)$ . Black line is  $\text{const.} \times (\rho_c - \rho)^{-n}$  with  $n = 1$ . Red rectangles are for Case III (one and two particle fragmentation), blue triangles are for Case IV [ $\Delta = 1, 2, \dots$  is discrete with fragmentation rate  $w(\Delta) = \exp(-\Delta)$ ] and red circles are for case V [ $\Delta > 0$  is continuous with fragmentation rate  $w(\Delta) = \exp(-\Delta)$ ]. Diffusion rate  $D(m) = 1$  throughout.

$w(\Delta = 2) = w_2$  and  $w(\Delta) = 0$  otherwise, Case IV. - a discrete-mass model with  $w(\Delta) = \exp(-\Delta)$  and Case V. - a continuous-mass model with  $w(\Delta) = \exp(-\Delta)$ . In these cases, in the absence of an analytical expression of  $\sigma^2(\rho)$ , we checked in simulations (see Fig. 4) that the variance near critical point indeed has the behaviour  $\sigma^2 \sim (\rho_c - \rho)^{-n}$ , with  $n = 1$ , which therefore leads to the same power-law exponent  $\tau = 5/2$ .

One can also define an asymmetric version of the CMAMs discussed above. In one dimension, there are some nontrivial spatial correlations and the above mean-field analysis fails to capture the mass fluctuations in the system. However, in higher dimensions, the above results qualitatively remain same also for the asymmetric mass transfer and is consistent with [24].

Interestingly, the exponent  $\tau = 5/2$  appears also in the distribution of particle numbers in ideal Bose gas in three dimensions (3D) near the critical point where Bose-Einstein condensation (BEC) occurs. This could be easily understood from the fact that particle-number fluctuation in the case of 3D Bose gas has the same critical behaviour  $\sigma^2(\rho) \sim (\rho_c - \rho)^{-n}$ , with  $n = 1$ , as in these ‘mean-field’ nonequilibrium systems having negligible spatial correlations. That, on a mean-field level, the nonequilibrium aggregation models belong to the universality class of equilibrium Bose gas in 3D, so far has not been realized.

It is quite instructive to consider a limiting case of Eq. 7 with  $n = 0$ ,  $\rho_c = \infty$  and  $g(\rho) \sim \rho^{1-\delta}$  at large density, i.e., the variance  $\sigma^2(\rho) \sim \rho^{1-\delta}$ , with  $\delta < -1$ , diverges algebraically at infinite density. As there is no singularity in the variance at any finite density, our analysis quite straightforwardly shows that condensation transition cannot occur, consistent with the past observations in the CMAM with mass-dependent diffusion [34]. Asymptotic scaling of the mass distribution can be obtained as follows. Using Eq. 3, we get  $\mu(\rho) \sim \rho^\delta$  (setting  $\alpha = 0$  without loss of any generality) and, consequently, Laplace transform of weight factor  $w_v(m)$ ,

$$\tilde{w}_v(s) \simeq a_0 + a_1 s^{1+1/\delta}, \quad (29)$$

immediately leading to mass distributions having a scaling form  $P_v(m) \propto m^{-\tau} \exp(\tilde{\mu}m) \equiv (m^*)^{-\tau} \Phi(m/m^*)$ . Here, the scaling function  $\Phi(x) = x^{-\tau} \exp(-x)$  with  $m^* \sim \rho^{-\delta}$  and power law exponent  $\tau = 2 + 1/\delta$  with  $1 < \tau < 2$  (as  $\delta < -1$ ), leading to a relation  $\delta(\tau - 2) = 1$ . The scaling form was numerically observed in [34]. Interestingly, the borderline case with  $\delta = -1$  generates gamma distributions, which are found in a broad class of mass transport processes [31] and have been also observed in a limiting case of conserved-mass aggregation models studied in Ref. [34].

## B. Nonconserved Mass Aggregation Models

In this section, we discuss a nonconserved version of the mass aggregation models where systems can exchange

mass, though *weakly*, with environment. In this case, in addition to the earlier defined two processes (1) diffusion and (2) fragmentation of masses, a particle now can be adsorbed with rate  $q$  and desorbed at a site with rate  $p$ , provided the site is occupied, where  $p, q \rightarrow 0$  (i.e., weak exchange) with the ratio  $r = q/p$  finite. Due to adsorption and desorption processes, total mass in the system is not conserved. This model is related to several models studied in the past for finite  $p$  and  $q$  [17, 19, 35, 36]. Interestingly, in the limit of  $p, q \rightarrow 0$ , mass fluctuation in a nonconserved model can be obtained from the occupation probability of a site in its conserved version (i.e.,  $p = q = 0$ ) [37, 38]. Let us first define, in the space of total mass  $M$ , a transition rate  $T_{M+1,M}$  from mass  $M$  to  $M + 1$ . In the steady state, the probability  $P(M)$  that the system has mass  $M$  satisfies a balance condition

$$P(M)T_{M+1,M} = P(M+1)T_{M,M+1}$$

where the mass distribution  $P(M)$  can be obtained as

$$P(M+1) = \left[ \prod_{M=0}^M \frac{T(M \rightarrow M+1)}{T(M+1 \rightarrow M)} \right] P(0). \quad (30)$$

As the ratio of transition rates can be written as

$$\frac{T_{M+1,M}}{T_{M,M+1}} = \frac{q}{p\mathcal{S}(\rho)}$$

where  $\mathcal{S}(\rho)$  is the occupation probability and  $\rho = M/V$ , the distribution function can be written, upto a normalization factor, as

$$P(M) \propto e^{\sum_M [\ln(q/p\mathcal{S})]} \simeq e^{-V \int_0^\rho d\rho [\mu(\rho) - \mu_0]} \quad (31)$$

where  $\mu_0 = \ln(q/p)$  an effective chemical potential and  $f(\rho) = \int d\rho \mu(\rho) = \int d\rho \ln \mathcal{S}(\rho)$  an effective free energy (canonical) density function. The steady state mass density as a function of adsorption to desorption ratio  $r = q/p$  can be obtained by minimizing the grand potential or the large deviation function for the density fluctuation  $h(\rho) = f(\rho) - \mu_0 \rho$ , leading to the relation  $\mathcal{S}(\rho) = r$  (for details, see Appendix C).

Till now, the analysis is exact. However, it may not always be possible to exactly calculate the occupation probability  $\mathcal{S}(\rho)$ . For the purpose of demonstration, let us proceed by considering a model with diffusion and fragmentation rate as in Case I. We obtain an approximate expression, obtained within mean-field theory, of the occupation probability (see Appendix C)

$$\mathcal{S}(\rho) = \frac{\rho(1-\rho)}{(1+\rho)}.$$

Then, Eq. 31 implies the subsystem mass distribution having a form  $P_v(m) \propto w_v(m) \exp(\mu m)$  and consequently a FR relation as in Eq. 3 follows. Then, for  $\rho < \rho_c$  or equivalently for  $r < r_c$ , one can immediately calculate the scaled variance as

$$\sigma^2(\rho) = \left( \frac{d\mu_0}{d\rho} \right)^{-1} = \frac{\rho(1-\rho)(1+\rho)}{(1-2\rho-\rho^2)}, \quad (32)$$

where critical density  $\rho_c = \sqrt{2} - 1$ . The variance in non-conserved case is different from that in the conserved-mass case, implying that the canonical and grand canonical ensembles are not equivalent [37, 38]. However, the nature of singularity in the variance remains the same near criticality where  $\sigma^2(\rho) \sim (\rho_c - \rho)^{-n}$  with  $n = 1$ . Therefore the additivity property leads to the same power law scaling in the single-site mass distribution  $P_1(m) \sim m^{-\tau} \exp(\tilde{\mu}m)$ , for large  $m$ , where  $\tau = 5/2$  and  $\tilde{\mu} = \mu_0 - \ln \mathcal{S}(\rho_c) = \ln(r/r_c)$  with  $r_c = \mathcal{S}(\rho_c)$ .

The above results are consistent with what was found, on the mean-field level, for general  $p$  and  $q$  in ‘in-out’ model [35] - a special case of the above nonconserved model with  $w = 0$ . One can interpret the results in the light of equilibrium BEC: The critical density signifies that, for  $r > r_c = \mathcal{S}(\rho_c) = 3 - 2\sqrt{2}$ , there is a condensate as in the BEC. In the grand-canonical setting (i.e., with no mass conservation), that would imply a phase with a diverging mass density, similar to the ‘Takayasu phase’ where mass density actually diverges. For  $p$  and  $q$  finite, form of the subsystem mass distribution as written in Eq. 31 remains the same, but only that the expression of  $\mathcal{S}(\rho)$ , due to the presence of spatial correlations, is different. However, the similarity with the BEC still persists.

#### IV. SUMMARY AND CONCLUDING PERSPECTIVE

In this paper, we argue that an additivity property can possibly explain why simple power-law scaling appears generically in nonequilibrium steady states with short-ranged correlations. We demonstrate that the existence of a fluctuation-response relation, a direct consequence of additivity, with a singular response function leads to power-law distributions with nontrivial exponents. The simplest form of the singularity, a simple pole, gives rise to the exponent 5/2, which was often observed in the past in apparently unrelated systems. We substantiate the claims by analytically calculating the response function, which diverges as critical point is approached, in paradigmatic nonequilibrium mass aggregation models and the corresponding single-site as well as subsystem mass distributions. Most remarkably, the analysis, being independent of dynamical rules in a particular system, equally extends to critical properties in equilibrium and nonequilibrium.

Thermodynamic characterization of phase coexistence in driven systems is a fundamental problem in statistical physics and has been addressed in the past [28, 29, 39–43], either numerically or analytically only for exactly known steady-states mostly having a product measure. From that perspective, it is quite encouraging that, even when steady-state weights are a priori *not known*, our analytical method not only gives insights into the steady-state structure but can also be applied to identify a chemical potential, which equalizes in the coexisting phases

and vanishing of which at the criticality gives rise to pure power laws.

Note that, in our formulation, the mass distribution functions, though approximate, have been calculated solely from the knowledge of the variance. This formulation is perhaps not surprising in equilibrium where free energy function (or entropy, for an isolated system) essentially determines fluctuation properties of a system. However, in nonequilibrium scenario, it is a-priori not clear that such equilibrium thermodynamic approach can indeed be applied in systems having a steady state with nontrivial spatial structure. Here, it is worth mentioning that one requires, in principle, all the moments to specify a probability distribution function. However, additivity property, provided it holds, puts a strong constraint on the mass distribution function  $P_v(m)$  through a fluctuation-response relation and thus helps to uniquely determine  $P_v(m)$ , only from the knowledge of the variance as a function of density.

We believe that our analysis, being based on a general thermodynamic principle, would be applicable in many other driven systems where phase coexistence is known to occur (e.g., in active matters [44, 45]). As a concluding remark, we mention that additivity property is expected to be quite generic for systems having short-ranged correlations and, therefore, it would be indeed interesting to actually verify additivity, through the predictions concerning fluctuations, on a case-by-case basis. Also, it remains to be seen whether the principle of additivity can be extended to systems having long-ranged spatial correlations, at least in the cases where the strength of these correlations are weak.

## V. ACKNOWLEDGEMENT

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## APPENDIX

In this Appendix, we provide the details of the calculations to obtain the mass distributions, using additivity property, in mass aggregation models (both conserved and nonconserved versions) which were studied over the last couple of decades. The generalized models introduced here cover some of those studied in the past and their variants [15, 16, 19–21, 24, 35]

## APPENDIX A: CALCULATION OF VARIANCE IN CONSERVED MASS AGGREGATION MODELS (CMAM)

We define here a class of conserved mass aggregation models (CMAM) on a one dimensional lattice with periodic boundary and calculate the variance of mass at a single site in the steady state, assuming that the additivity property (Eq. 1) holds. For, simplicity, we consider only the discrete-mass cases.

The mass at each site undergoes either diffusion (where whole of the mass is transferred to either of neighbouring sites) or chipping, with certain transition rates; in the models considered below, there are two types of chipping process. The diffusing mass or the chipped-off mass are coalesced with the mass at either of the neighbouring sites with a pre-assigned rates. In this process, the total mass of the system is conserved.

Provided a site  $i$  is occupied, particles hop to either of the nearest neighbour sites according to the transition rates specified below.

*A. Diffusion with rate 1:* All particles at a site  $i$  hop with rate 1 to left or right, i.e.,  $m_i \rightarrow 0$  and  $m_{i\pm 1} \rightarrow m_{i\pm 1} + m_i$ .

*B. Chipping with rate  $w_1$ :* This chipping process involves a particle at site  $i$  being chipped off and thrown to left or right neighbour, i.e.,  $m_i \rightarrow (m_i - 1)$  and  $m_{i\pm 1} \rightarrow m_{i\pm 1} + 1$ .

*C. Chipping with rate  $w_2$ :* This chipping process involves  $m_i - 1$  particles going to either left or right neighbour and the rest of the particles remaining at site  $i$ , i.e.,  $m_i \rightarrow 1$  and  $m_{i\pm 1} \rightarrow m_{i\pm 1} + m_i - 1$ .

Assuming transition rates are Poissonian, we have the following stochastic update rules where mass  $m_i(t + dt)$  at time  $t + dt$  takes a particular value, depending on mass  $m_i(t)$  at time  $t$ , with certain probabilities as shown below.

**Loss terms at site  $i$ :**

$$m_i(t + dt) = \begin{cases} \text{value:} & \text{probability:} \\ 0 & \frac{dt}{dt} \\ m_i(t) - 1 + \delta_{m_i(t),0} & w_1 dt \\ 1 - \delta_{m_i(t),0} & w_2 dt. \end{cases}$$

**Gain terms from  $(i - 1)^{th}$  site:**

$$m_i(t + dt) = \begin{cases} \text{value:} & \text{prob.:} \\ m_i(t) + m_{i-1}(t) & \frac{dt}{2}, \\ m_i(t) + 1 - \delta_{m_{i-1}(t),0} & w_1 \frac{dt}{2}, \\ m_i(t) + m_{i-1}(t) - 1 + \delta_{m_{i-1}(t),0} & w_2 \frac{dt}{2}. \end{cases}$$



**Gain terms from  $(i+1)^{th}$  site:**

$$m_i(t+dt) = \begin{cases} \text{value:} & \text{prob.:} \\ m_i(t) + m_{i+1}(t) & \frac{dt}{2}, \\ m_i(t) + 1 - \delta_{m_{i+1}(t),0} & w_1 \frac{dt}{2} \\ m_i(t) + m_{i+1}(t) - 1 + \delta_{m_{i+1}(t),0} & w_2 \frac{dt}{2}. \end{cases}$$

**Mass remains unchanged at site  $i$ :**

$$m_i(t+dt) = \begin{cases} \text{value:} & \text{prob.:} \\ m_i(t) & (1 - 2dt - 2w_1dt - 2w_2dt). \end{cases}$$


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Now we define the occupation probability  $\langle(1 - \delta_{m_j,0})\rangle = \mathcal{S}(\rho)$ , i.e., the probability that a site is occupied. We deal with steady-state averages throughout. We assume that the additivity property (as in Eq. 1) is valid at single site level and therefore  $n$ -point ( $n \geq 2$ ) correlation factorizes.

**$n$ -th moment equation:** The time evolution of  $n$ -th moment  $\langle m_i^n \rangle$  can be written as

$$\begin{aligned} \langle m_i^n(t+dt) \rangle &= \langle m_i^n(t) \rangle = \langle [m_i(t) - 1 + \delta_{m_i(t),0}]^n \rangle w_1 dt + \langle [m_i(t) + m_{i-1}(t)]^n \rangle \frac{dt}{2} + \langle [1 - \delta_{m_i(t),0}]^n \rangle w_2 dt \\ &+ \langle [m_i(t) + 1 - \delta_{m_{i-1}(t),0}]^n \rangle w_1 \frac{dt}{2} + \langle [m_i(t) + m_{i-1}(t) - 1 + \delta_{m_{i-1}(t),0}]^n \rangle w_2 \frac{dt}{2} + \langle [m_i(t) + m_{i+1}(t)]^n \rangle \frac{dt}{2} \\ &+ \langle [m_i(t) + 1 - \delta_{m_{i+1}(t),0}]^n \rangle w_1 \frac{dt}{2} + \langle [m_i(t) + m_{i+1}(t) - 1 + \delta_{m_{i+1}(t),0}]^n \rangle w_2 \frac{dt}{2} \\ &+ \langle m_i^n(t) \rangle (1 - 2dt - 2w_1dt - 2w_2dt), \end{aligned} \quad (33)$$


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which, in the steady state where  $\langle m_i^n(t+dt) \rangle = \langle m_i^n(t) \rangle$ , gives a BBGKY hierarchy where  $n$ -point correlations are coupled to  $(n+1)$ -point correlations. To get a closed set of equations for the moments, we use the factorization property of  $n$ -point correlations. As mentioned in the paper, the mass distributions are *solely* obtained from the response function (or the variance of the mass distribution) and therefore we are interested in only calculating the variance, or equivalently the second moment, which can be done as follows.

**2<sup>nd</sup> moment equation:** If we put  $n = 2$  in the above equation, the second moment  $\langle m_i^2 \rangle$  however cancels out from the above equation. Using factorization of two-point correlation, i.e.,  $\langle m_i m_j \rangle \approx \rho^2$  for  $i \neq j$ , we get an expression for the occupation probability  $\mathcal{S}(\rho)$  as a function of mass density  $\rho$ ,

$$\rho^2(1 + w_2) = w_+(\rho - \mathcal{S}) - w_-\rho\mathcal{S}, \quad (34)$$

where  $w_{\pm} = w_1 \pm w_2$ . This gives

$$\mathcal{S}(\rho) = \frac{w_+\rho - (1 + w_2)\rho^2}{w_+ + w_-\rho}. \quad (35)$$

**3<sup>rd</sup> moment equation:** Similarly, for  $n = 3$ , we obtain an equation where the third moment  $\langle m_i^3 \rangle$  cancels out and we actually get, using factorization of both two-point and three-point correlation, a relation for the second moment

$$\langle m^2 \rangle = \rho \frac{w_+(1 + \mathcal{S}) - 2w_2\rho}{w_+ - 2(1 + w_2)\rho - w_- \mathcal{S}} \quad (36)$$

Using the expression of occupation probability in Eq. 35, we obtain

$$\langle m^2 \rangle = \rho \frac{w_+^2 + 2w_+w_-\rho - (w_+ + 3w_1w_2 - w_2^2)\rho^2}{w_+^2 - 2w_+(1 + w_2)\rho - w_-(1 + w_2)\rho^2} \quad (37)$$

which leads to the desired expression of the variance as a function of density,

$$\begin{aligned} \sigma^2(\rho) &= \frac{w_+^2\rho + w_+(w_1 - 3w_2)\rho^2}{w_+^2 - 2w_+(1 + w_2)\rho - w_-(1 + w_2)\rho^2} \\ &+ \frac{(w_+ - w_1w_2 + 3w_2^2)\rho^3 + w_-(1 + w_2)\rho^4}{w_+^2 - 2w_+(1 + w_2)\rho - w_-(1 + w_2)\rho^2} \end{aligned} \quad (38)$$

The variance  $\sigma^2(\rho)$  has a *singularity* at  $\rho = \rho_c$ , i.e., it diverges at a critical density  $\rho = \rho_c$ , which can be obtained by putting the denominator of Eq. 38 zero and solving

$$w_+^2 - 2w_+(1 + w_2)\rho_c - w_-(1 + w_2)\rho_c^2 = 0. \quad (39)$$

This gives a *simple pole* at the critical density

$$\rho_c = \frac{w_+}{w_-} \left( \sqrt{1 + \frac{w_-}{1 + w_2}} - 1 \right). \quad (40)$$

Nonequilibrium free energy function can be calculated by integrating nonequilibrium chemical potential w.r.t. density  $\rho$ ,

$$\mu(\rho) = \frac{df}{d\rho} \Rightarrow f(\rho) = \int \mu(\rho) d\rho. \quad (41)$$

The function  $\lambda_v(s) = -\ln \tilde{w}(s)$ , which is the Legendre transform of the free energy density  $f(\rho)$ , can be obtained as given below,

$$\lambda_v(s) = v[f(\rho^*) + s\rho^*], \quad (42)$$

where  $\rho^*$  is the solution of

$$s = -\mu(\rho^*). \quad (43)$$

### APPENDIX B: CALCULATION OF MASS DISTRIBUTION IN THE CONSERVED MASS AGGREGATION MODEL

Here we provide the essential steps of the calculations to obtain single-site (i.e.,  $v = 1$ ) mass distribution function  $P_1(m) \propto w_1(m) \exp[\mu(\rho)m]$  where  $w_1(m)$  is the single-site weight factor and  $\mu(\rho)$  is a chemical potential. We first analyse the behaviour of  $\lambda_1(s)$  near the singularity at  $s = s_c$  by expanding  $\mu(\rho)$  and  $f(\rho)$  near critical density in the power series of  $\rho - \rho_c$  where  $\rho - \rho_c < 0$  is small,

$$\begin{aligned} \mu(\rho) &= \mu(\rho_c) + \frac{\mu''(\rho_c)}{2}(\rho - \rho_c)^2 + \dots \\ f(\rho) &= f(\rho_c) + \mu(\rho_c)(\rho - \rho_c) + \frac{f'''(\rho_c)}{3!}(\rho - \rho_c)^3 + \dots \end{aligned} \quad (44)$$

where we have used Eq. 41 and  $\mu'(\rho_c) = f''(\rho_c) = 0$  (see Fig. 1). Using Eq. 43 in Eq. 44 and substituting  $s + \mu(\rho_c) \simeq -\mu''(\rho_c)(\rho^* - \rho_c)^2/2$ , we get

$$(\rho^* - \rho_c) = -\sqrt{\frac{2}{|\mu''(\rho_c)|}}(s - s_c)^{1/2} \quad (45)$$

where  $s_c = -\mu(\rho_c)$  and  $\mu''(\rho_c) < 0$ . Therefore  $\lambda_1(s) = f(\rho^*) + s\rho^*$  near  $s = s_c$ , in the leading order of  $(s - s_c)$ , can be approximated as

$$\begin{aligned} \lambda_1(s) &\simeq \left[ f(\rho_c) - s_c(\rho^* - \rho_c) + \frac{f'''(\rho_c)}{3!}(\rho^* - \rho_c)^3 \right] + s\rho^* \\ &= \lambda_1(s_c) + \rho^*(s - s_c) + \frac{f'''(\rho_c)}{3!}(\rho^* - \rho_c)^3 \\ &= \left[ a_0 + a_1(s - s_c) + a_2(s - s_c)^{3/2} \right] \end{aligned} \quad (46)$$

where  $a_0 = \lambda_1(s_c) = f(\rho_c) + s_c\rho_c$ ,  $a_1 = \rho_c$  and  $a_2 = -(2/3)\sqrt{2/|\mu''(\rho_c)|}$ . The inverse Laplace transform of the weight factor  $w_1(m)$  can be written as

$$\tilde{w}_1(s) = e^{-\lambda_1(s)} \simeq e^{-a_0} [1 - a_1(s - s_c) - a_2(s - s_c)^{3/2}] \quad (47)$$

which, for  $m \gg 1$ , translates into

$$w_1(m) \sim \frac{e^{s_c m}}{m^{5/2}}. \quad (48)$$

Consequently the mass distribution can be written as

$$P_1(m) \sim \frac{e^{s_c m}}{m^{5/2}} e^{\mu(\rho)m} = \frac{e^{-(\alpha + \mu_0(\rho_c))m}}{m^{5/2}} e^{(\mu_0(\rho) + \alpha)m} \quad (49)$$

$$P_1(m) \sim \frac{1}{m^{5/2}} e^{[\mu_0(\rho) - \mu_0(\rho_c)]m}. \quad (50)$$

Note that effective chemical potential  $\tilde{\mu}(\rho) = \mu_0(\rho) - \mu_0(\rho_c)$  is zero at the critical density  $\rho_c = (\sqrt{2} - 1)$ . The mass distribution in Eq. 50 is precisely what was found in [19] at  $\rho = \rho_c$  and describes the simulation data remarkably well (see Fig. 1).

### APPENDIX C: CALCULATION OF MASS DISTRIBUTION IN THE ABSENCE OF MASS CONSERVATION

As shown in the paper, the probability distribution function  $P(M)$  of total mass  $M$  can be written, up to a normalization factor, as

$$P(M) = \text{const.} \times e^{-V \int_0^\rho d\rho [\mu(\rho) - \mu_0]} \quad (51)$$

Now, if we assume that the joint mass distribution  $\mathcal{P}[\{m_i\}]$  has a product form on single-site level ( $v = 1$ ), i.e., product of single-site mass distribution function  $p(m_i)$ ,

$$\mathcal{P}[\{m_i\}] = \prod_{i=1}^V p(m_i), \quad (52)$$

the probability distribution function  $P(M)$  of mass  $M$  in the system can be written as

$$P(M) = \prod_{i=1}^V \left[ \int dm_i p(m_i) \right] \delta \left( M - \sum_i m_i \right). \quad (53)$$

From the Laplace transform  $\tilde{P}(s) = \int dM P(M) \exp(-sM) = [\tilde{p}(s)]^V$  of the mass distribution  $P(M)$ , the Laplace transform  $\tilde{p}(s) = \int dm_i p(m_i) \exp(-sm_i)$  of single-site mass distribution  $p(m)$  can be written as

$$\tilde{p}(s) = \text{const.} \times e^{-\lambda_1(s)}, \quad (54)$$

where

$$\lambda_1(s) = \inf_\rho [h(\rho) + s\rho]. \quad (55)$$

Here we have used inverse transform

$$\tilde{P}(s) = \text{const.} \times \int d\rho e^{-V[h(\rho) + s\rho]}, \quad (56)$$

which has been obtained from Eq. 51 and where grand potential or the large deviation function for density fluctuation  $h(\rho) = f(\rho) - \mu_0\rho = \int_0^\rho [\mu(\rho) - \mu_0] d\rho$  and chemical potential  $\mu(\rho) = \ln \mathcal{S}(\rho) = \ln[\rho(1-\rho)/(1+\rho)]$ , as given in the paper. Note that the function  $\mathcal{S}(\rho)$  is the occupation probability in the conserved mass aggregation model and has been obtained by putting  $w_1 = 1$  and  $w_2 = 0$  in Eq. 35.

Now the function  $\lambda_1(s)$ , Legendre transform of grand potential  $h(\rho)$ , can be written as

$$\lambda_1(s) = h(\rho^*) + s\rho^*, \quad (57)$$

where  $\rho^*$  is the root of the equation  $d[h(\rho) + s\rho]/d\rho = 0$  or  $\mu(\rho^*) - \mu_0 + s = 0$ , i.e.,  $\rho^*$  is the root of

$$\ln \left[ \frac{\rho^*(1-\rho^*)}{1+\rho^*} \right] = \mu_0 - s. \quad (58)$$

The critical density is obtained by putting scaled variance as  $\sigma^2(\rho) = (d\mu_0/d\rho)^{-1} = \infty$  or  $1/\sigma^2(\rho) = 0$ ,

$$\frac{(1 - 2\rho_c - \rho_c^2)}{\rho_c(1 - \rho_c)(1 + \rho_c)} = 0, \quad (59)$$

and thus  $\rho_c = \sqrt{2} - 1$ . In the macrostate (most probable state), we have  $\mathcal{S}(\rho) = r$ , implying that the critical density is related to the ratio  $r = q/p$  through  $\mathcal{S}(\rho_c) = r_c$ . To obtain the large-mass behaviour, we expand  $\mu(\rho)$  around  $\rho = \rho_c$ ,

$$\mu(\rho) = \mu(\rho_c) + \frac{\mu''(\rho_c)}{2}(\rho - \rho_c)^2, \quad (60)$$

to obtain

$$(s - s_c) \simeq \frac{|\mu''(\rho_c)|}{2}(\rho^* - \rho_c)^2, \quad (61)$$

$$\lambda_1(s) \simeq a_0 + a_1(s - s_c) + a_2(s - s_c)^{3/2}, \quad (62)$$

in leading order in  $(\rho^* - \rho_c)$  where  $s_c = \mu_0 - \mu(\rho_c)$ , leading to the desired result in the paper,

$$p(m) \sim \frac{1}{m^{5/2}} e^{s_c m} = \frac{1}{m^{5/2}} e^{[\mu_0 - \mu(\rho_c)]m}. \quad (63)$$

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- [1] P. Bak, *How Nature Works: The Science of Self-Organised Criticality*, Copernicus Press, New York (1996);
- [2] A. E. Scheidegger, Int. Assoc. Sci. Hydrol. Bull. **12**, 15 (19677).
- [3] S. K. Friedlander, *Smoke, Dust and Haze* (Wiley Interscience, New York, 1977).
- [4] F. Family and P. Meakin, Phys. Rev. Lett. **61**, 428 (1988). R. D. Narhe *et. al.*, Phys. Rev. Lett. **86**, 1570 (2001). A. A. Lushnikov, Phys. Rev. Lett. **93**, 198302 (2004).
- [5] R. M. Ziff, E. M. Hendriks, and M. H. Ernst, Phys. Rev. Lett. **49**, 593 (1982). R. M. Ziff, J. Stat. Phys. **23** 241 (1980).
- [6] O. Peters, C. Hertlein, and K. Christensen, Phys. Rev. Lett. **88**, 018701 (2002). R. Dickman, Phys. Rev. Lett. **90**, 108701 (2003).
- [7] *Statistical models for the fracture of disordered media*, Ed. H.J. Herrmann and S. Roux, (North Holland, Amsterdam, 1990).
- [8] G. K. Zipf, *Human Behavior and the Principle of Least Effort* (Addison-Wesley Press, Cambridge, 1949).
- [9] M. Marsili and Y. C. Zhang, Phys. Rev. Lett. **80**, 2741 (1998). J. Ke *et. al.*, Phys. Rev. Lett., **97**, 028301 (2006).
- [10] H. E. Stanley, V. Plerou, and X. Gabaix, Physics A **387** 3967 (2008).
- [11] J. M. Beggs and D. Plenz, J. Neurosci **23**, 11167 (2003); X. Chen *et. al.*, Phys. Rev. Lett. **108**, 148101 (2012). D. Krotov, J. O. Dubuis, T. Gregor and W. Bialek, PNAS **111**, 3683 (2014); C. Furusawa and K. Kaneko, Phys. Rev. Lett. **108**, 208103 (2012); J. Hidalgo *et. al.*, PNAS **111**, 10095 (2014).
- [12] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
- [13] A. Vespignani, R. Dickman, M. A. Munoz, and S. Zapperi, Phys. Rev. Lett. **81**, 5676 (1998).
- [14] P. K. Mohanty and D. Dhar, Phys. Rev. Lett. **89**, 104303 (2002).
- [15] R. D. Vigil, R. M. Ziff, and B. Lu, Phys. Rev. B **38**, 942 (1988).
- [16] P. L. Krapivsky and S. Redner, Phys. Rev. E **54**, 3553 (1996).
- [17] H. Takayasu, Phys. Rev. Lett. **63**, 2563 (1989).
- [18] S. N. Majumdar and C. Sire, Phys. Rev. Lett. **71**, 3729 (1993).
- [19] S. N. Majumdar, S. Krishnamurthy, and M. Barma, Phys. Rev. Lett. **81**, 3691 (1998).
- [20] G. P. Shrivastav, V. Banerjee, and S. Puri, Phase Transitions **83**, 140 (2010).
- [21] S. N. Majumdar, S. Krishnamurthy, and M. Barma, J. Stat. Phys. **99** 1 (2000).
- [22] D. Dhar and R. Ramaswamy, Phys. Rev. Lett. **63**, 1659 (1989).
- [23] R. Rajesh and S. N. Majumdar, Phys. Rev. E **63**, 036114 (2001).
- [24] R. Rajesh and S. Krishnamurthy, Phys. Rev. E **66**, 046132 (2002).
- [25] W. H. Stockmayer, J. Chem. Phys. **11**, 45 (1943). P. J. Flory, *Principles of Polymer Chemistry* (Cornell Univ. Press, Ithaca, 1953).
- [26] M. Kardar, *Statistical Physics of Particles*, Cambridge University Press (2007).
- [27] G. L. Eyink, J. L. Lebowitz, H. Spohn, J. Stat. Phys **83**, 385 (1996).
- [28] E. Bertin, O. Dauchot, and M. Droz, Phys. Rev. Lett. **96**, 120601 (2006).
- [29] E. Bertin, K. Martens, O. Dauchot, and M. Droz, Phys. Rev. E **75**, 031120 (2007).
- [30] P. Pradhan, C. P. Amann, and U. Seifert, Phys. Rev. Lett. **105**, 150601 (2010).
- [31] S. Chatterjee, P. Pradhan and P. K. Mohanty, Phys. Rev. Lett. **112**, 030601 (2014).
- [32] H. Touchette, Physics Reports **478**, 1 (2009).
- [33] For example, a normalizable distribution  $P_v(m) \sim \frac{1}{m_1^*} \Phi_1(\frac{m}{m_1^*}) + \frac{1}{(m_2^*)^2} \Phi_2(\frac{m}{m_2^*})$ , near criticality, can have a finite mean  $\langle m \rangle = \rho$  but diverging second moment  $\langle m^2 \rangle$  for a suitable choice of a finite  $m_1^*(\rho)$  and a diverging  $m_2^*(\rho) \sim (\rho - \rho_c)^{-n}$ ; a simple choice could be  $\Phi_{1,2}(x) = \exp(-\kappa_{1,2} x^{\delta_{1,2}})$  with  $\kappa_{1,2}, \delta_{1,2} > 0$ .
- [34] R. Rajesh, D. Das, B. Chakraborty, and M. Barma, Phys. Rev. E **66**, 056104 (2002).
- [35] S. N. Majumdar, S. Krishnamurthy, and M. Barma, Phys. Rev. E **61**, 6337 (2000).
- [36] R. Rajesh, Phys. Rev. E **69**, 036128 (2004).
- [37] O. Cohen and D. Mukamel, Phys. Rev. Lett. **108**, 060602 (2012).
- [38] S. Chatterjee, P. Pradhan, and P. K. Mohanty, Phys. Rev. E **91**, 062136 (2015).
- [39] S. N. Majumdar, M.R. Evans, and R. K. P. Zia, Phys. Rev. Lett. **94**, 180601 (2005).

- [40] M. R. Evans, S. N. Majumdar, and R. K. P. Zia, J. Phys. A: Math. Gen. **37**, 275 (2004).
- [41] R. K. P. Zia, M. R. Evans, and S. N. Majumdar, J. Stat. Mech.: Theor. Exp. **L10001** (2004).
- [42] P. Pradhan and U. Seifert, Phys. Rev. E **84**, 051130 (2011).
- [43] R. Dickman, Phys. Rev. E **90**, 062123 (2014).
- [44] M. C. Marchetti, J. F. Joanny, S. Ramaswamy, T. B. Liverpool, J. Prost, M. Rao, and R. A. Simha, Rev. Mod. Phys. **85**, 1143 (2013).
- [45] S. Chakraborti, S. Mishra and P. Pradhan (unpublished).